

Branes from Matrices

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Various aspects of branes in the recently proposed matrix model for M theory are discussed. A careful analysis of the supersymmetry algebra of the matrix model uncovers some central charges which can be activated only in the large N limit. We identify the states with non-zero charges as branes of different dimensions.

1. Introduction

The matrix model approach to M theory [1] has successfully passed a number of consistency tests [1-6]. One outstanding problem that remains to be solved is the proper description of five branes in the matrix model. Berkooz and Douglas [2] have introduced five branes wrapped around the longitudinal direction of the light cone frame by adding degrees of freedom to the original model of [1]. The purpose of the present paper is to define a general formalism for the study of Bogolmonyi Prasad Sommerfield (BPS) p -branes in the matrix model.

The eleven dimensional supersymmetry (SUSY) algebra

$$\{Q_\alpha, Q_\beta\} = 2P^\mu \gamma_{\mu\alpha\beta} + 2Z^{\mu_1\mu_2} \gamma_{\mu_1\mu_2\alpha\beta} + 2Z^{\mu_1\cdots\mu_5} \gamma_{\mu_1\cdots\mu_5\alpha\beta} \quad (1.1)$$

($\mu_i = 0, \dots, 10$; $\alpha, \beta = 1, \dots, 32$) includes two central charges: a two brane charge $Z^{\mu_1\mu_2}$ and a five brane charge $Z^{\mu_1\cdots\mu_5}$. These charges are Lorentz tensors and as such their existence seems to violate the Coleman-Mandula theorem. A contradiction is avoided by noticing that when these charges do not vanish they are infinite. This infinity has a simple physical interpretation. The charged objects are branes which in flat eleven dimensional space have infinite volume. The total charge is infinite but the charge per unit volume is finite. A simple way of handling these infinities is to assume that space is compactified on a very large but finite torus. Then, the charges are all finite. Since we keep the time non-compact, $Z^{0\mu_2} = Z^{0\mu_2\cdots\mu_5} = 0$. Equivalently, this result follows from the fact that the Lorentz indices of the conserved currents associated with these central charges $j_{\mu_1\mu_2\mu_3}$ and $j_{\mu_1\cdots\mu_6}$ are antisymmetric. The charges are given by integrals over space of $j_{\mu_1\mu_2 0}$ and $j_{\mu_1\cdots 0}$ and therefore $Z^{0\mu_2} = Z^{0\mu_2\cdots\mu_5} = 0$.

In the light cone frame only $SO(9) \subset SO(10,1)$ is manifest – the 11 coordinates x^μ become x^\pm, x^a ($a = 1\dots 9$) and the 32 supercharges become a pair of 16 supercharges Q_α ($\alpha = 1\dots 16$) which we will refer to as dynamical supercharges and \tilde{Q}_α which are kinematical. The supersymmetry algebra is

$$\begin{aligned} \{\tilde{Q}_\alpha, \tilde{Q}_\beta\} &= 2P^+ \delta_{\alpha\beta} \\ \{Q_\alpha, \tilde{Q}_\beta\} &= 2P^a \gamma_{a\alpha\beta} + 2Z^{a_1a_2} \gamma_{a_1a_2\alpha\beta} + 2Z^{a_1\cdots a_5} \gamma_{a_1\cdots a_5\alpha\beta} \\ \{Q_\alpha, Q_\beta\} &= 4P^- \delta_{\alpha\beta} + 2Z^a \gamma_{a\alpha\beta} + 2Z^{a_1\cdots a_4} \gamma_{a_1\cdots a_4\alpha\beta} \end{aligned} \quad (1.2)$$

In comparing (1.2) with (1.1) note that we have rescaled the supercharges by numerical constants. More important is the way the central charges $Z^{\mu_1\mu_2}$ and $Z^{\mu_1\cdots\mu_5}$ are handled. Since the time in the light cone frame is x^+ we have set to zero the charges with a component along $\mu = +$. The charges with components along $\mu = -$ are denoted by Z^a

and Z^{abcd} . They are activated by two branes and five branes which are stretched along the longitudinal direction. Clearly, to keep them finite we must compactify this direction on a circle of radius R and then they both scale like R .

From this algebra we can distinguish three types of BPS states in the infinite momentum frame (IMF).

1. Purely transverse membranes and fivebranes. For these, half of the SUSYs are preserved. They are linear combinations of the kinematical and dynamical generators. The IMF energy of these states is proportional to the square of the central charge $(Z^{ab})^2$ or $(Z^{abcde})^2$. This corresponds to the fact that if the transverse directions are compactified, these excitations propagate as particles. For such states the IMF energy (for zero transverse momentum) is proportional to the mass squared divided by the longitudinal momentum P^+ . Thus, the corresponding brane tensions are proportional to the central charge. The authors of [7], who were the first to demonstrate that the matrix model describes membranes, showed that membrane energies indeed scale like the inverse longitudinal momentum in the matrix model. This scaling is somewhat implicit in their formalism, because they did not make the identification of the longitudinal momentum with the rank of the matrices which was proposed in [1]. It follows from the relation between matrix commutators and Poisson brackets on the membrane volume.
2. Branes wrapped around the longitudinal direction. These have nonzero values of Z^a or Z^{abcd} , with all other central charges vanishing. They preserve one quarter of the SUSYs. All kinematical generators are broken and half the dynamical ones are preserved. The energy of such a state is simply proportional to the central charge. Furthermore it scales like a constant in the $P^+ \rightarrow \infty$ limit. A wrapped object can carry momentum in the direction of the circle on which it is wrapped only if it contains some internal excitation which breaks translation invariance around the circle. As the momentum is scaled to infinity, the internal excitation energy also goes up, so that the energy in the IMF does not scale like $1/P^+$.¹ It is somewhat remarkable that the lightcone SUSY algebra automatically takes this dynamical fact into account. This

¹ An explicitly calculable example of this phenomenon arises in first quantized string theory. The energy E of a state satisfies $E^2 = \frac{1}{2}P_L^2 + \frac{1}{2}P_R^2 + N_L + N_R$ where P_L and P_R are given in terms of the momentum P and the winding L as $P_L = P + L$ and $P_R = P - L$. If L in some direction is not zero and we let P in this direction go to infinity, the level matching condition $\frac{1}{2}P_L^2 - \frac{1}{2}P_R^2 + N_L - N_R = 2PL + N_L - N_R = 0$ implies that $|N_L - N_R|$ goes to infinity like $|P|$. Therefore, the “mass” square $m^2 = E^2 - P^2$ is not a constant in this limit but grows linearly with $|P|$, so that the IMF Hamiltonian $E - |P|$ goes to a constant.

internal excitation is also the source of the breaking of the extra half of the SUSY generators.

3. Branes wrapped around the longitudinal direction which preserve half the supersymmetry. As an example consider a brane with nonzero values for both the two and the four brane charges, related by the formula (for a particular choice of orientation of the solution) $P^+ Z^{1234} = Z^{12} Z^{34}$. This preserves half of the SUSYs, again a combination of the kinematical and dynamical generators. It is a longitudinal five brane, with two orthogonal infinite stacks of two branes embedded in it. If we insist that the four brane charge is finite, then the product of two brane charges must scale like the longitudinal momentum.

In terms of the fields in the low energy supergravity Lagrangian the supercharges appear as integrals of total derivatives, and are nonvanishing only on topologically nontrivial field configurations. In relating them to the matrix model [1] our guiding philosophy is that the analogous notion is that of commutators of infinite matrices which have nonzero trace. This idea, already hinted at in [1], is closely related to an interpretation of nonvanishing central charges as topological objects (integrals of total derivatives).

The plan of the rest of this paper is as follows: In the next section we compute the commutator of supercharge densities in the matrix model and show that it contains terms which may be interpreted as components of the membrane and five brane charge densities. We compare this to a computation of membrane supercharge densities.

In section 3 we find all static classical configurations of the matrix model which preserve half of the supersymmetries. Apart from the original membrane discovered in [1], these are p -brane configurations with infinite stacks of orthogonal membranes embedded in them. The simplest of these extends in four transverse directions. We show that it satisfies a BPS formula with nonzero values of the five brane charge, and may thus be interpreted as a five brane wrapped around the longitudinal dimension. It differs from the Berkooz-Douglas fivebrane, in that it carries membrane charge as well. The energy of the configuration scales in a manner consistent with this interpretation. Other configurations, with extent in 6 and 8 transverse dimensions, do not seem to have a conventional M theory interpretation. However, it may be that they correspond to D-brane excitations of string theory and only make sense after compactification to 10 dimensions. We show (in section 4) that the Lagrangian for small fluctuations around these configurations indeed reduces to the world volume Lagrangian of a D-brane. We note however that the tensions (energy densities) of these branes scale to infinity in the large N limit with eleven noncompact dimensions. We have not found a satisfactory interpretation of this result. Finally, we exhibit classical matrix configurations describing the longitudinal five brane with no membrane charge.

In section 4 we compute the small fluctuations around all of these configurations and verify that they include the collective coordinates implied by the p -brane interpretation. We also show how enhanced gauge symmetry arises when parallel two branes are brought together.

In the conclusions we point out that the matrix model supercharge density algebra does not have a term corresponding to the purely transverse components of the five brane charge density. We suggest that this may be a defect of the light cone gauge, related to the fact that purely transverse D-branes cannot be constructed in light cone gauge perturbative string theory. We present arguments which suggest that the transverse fivebrane cannot be a classical solution of the matrix model.

As this paper was being written, a revised version of [4] appeared, which also discusses the four brane charge and the self-dual solution.

2. The Supercharge Density Algebra

In this section we will construct the matrix analog of the supersymmetry algebra. The analog of integration over the membrane volume is taking the trace of a matrix. The SUSY generators are written as traces of products of matrices, and it is natural to define the untraced products to be the supercharge densities. As with any density algebra, we are free to add “improvement terms” to the densities, which do not affect the traced charges, at least for a large class of configurations of the system. For the present computation we will content ourselves with a minimal improvement which insures that the supercharge densities are hermitian matrices. This does not affect the supercharges for finite N , and the improvement also vanishes in the smooth membrane approximation to the infinite N system.

We use the conventions for nine dimensional gamma matrices

$$\begin{aligned}\{\gamma^a, \gamma^b\} &= 2\delta^{ab} \\ \gamma^{ab} &= \frac{1}{2}[\gamma^a, \gamma^b]\end{aligned}\tag{2.1}$$

and the other $\gamma^{ab\dots}$ normalized similarly. Recall the symmetry properties in the spinor indices $\delta_{(\alpha\beta)}$, $\gamma_{(\alpha\beta)}^a$, $\gamma_{[\alpha\beta]}^{ab}$, $\gamma_{[\alpha\beta]}^{abc}$, $\gamma_{(\alpha\beta)}^{abcd}$. We need the identity

$$I_{\alpha\beta\alpha'\beta'}^b = \gamma_{\beta\beta'}^a \gamma_{\alpha\alpha'}^{ab} + \gamma_{\beta\beta'}^{ab} \gamma_{\alpha\alpha'}^a + (\alpha \leftrightarrow \beta) = 2(\gamma_{\alpha'\beta'}^b \delta_{\alpha\beta} - \gamma_{\alpha\beta}^b \delta_{\alpha'\beta'}).\tag{2.2}$$

As in [1] we study the Lagrangian

$$\mathcal{L} = \text{Tr } L\tag{2.3}$$

where the Lagrangian density L is the $N \times N$ matrix

$$\begin{aligned} L &= \frac{1}{2R}(D_0 X^a)^2 + \theta^\alpha D_0 \theta^\alpha + \frac{R}{4}[X^a, X^b]^2 + \frac{iR}{2}[\theta^\beta, [X^a, \theta^\alpha]]\gamma_{\alpha\beta}^a \\ D_0 X^a &= \partial_0 X^a - i[A_0, X^a] \\ D_0 \theta^\alpha &= \partial_0 \theta^\alpha - i[A_0, \theta^\alpha]. \end{aligned} \quad (2.4)$$

θ^α ($\alpha = 1, \dots, 16$), X^a ($a = 1, \dots, 10$) and A_0 are hermitian $N \times N$ matrices and R is the radius of the longitudinal direction. The commutators in (2.4) are commutators of these matrices. The last term in L differs from the standard way of writing L by an anticommutator of odd variables. For finite matrices this does not affect its trace, \mathcal{L} . It was added here to make L hermitian (we use the convention that for u, v odd Grassman numbers $(uv)^* = u^* v^*$). Matrices have upper and lower $SU(N)$ indices and are multiplied according to

$$(AB)_i^j = A_i^k B_k^j. \quad (2.5)$$

The Hamiltonian corresponding to this Lagrangian in the $A_0 = 0$ gauge is

$$\mathcal{H} = R \text{Tr } h \quad (2.6)$$

with the hermitian Hamiltonian density

$$h = \frac{1}{2}P^2 - \frac{1}{4}[X^a, X^b]^2 - \frac{i}{2}[\theta^\alpha, [X^b, \theta^\beta]]\gamma_{\alpha\beta}^b. \quad (2.7)$$

The Dirac brackets (DB) are

$$\begin{aligned} [X_i^{aj}, P_k^{bl}]_{DB} &= \delta^{ab} \delta_i^l \delta_k^j \\ \{\theta_i^{\alpha j}, \theta_k^{\beta l}\}_{DB} &= \frac{1}{2} \delta^{\alpha\beta} \delta_i^l \delta_k^j. \end{aligned} \quad (2.8)$$

We note that in the computation which we will perform, no quantum mechanical operator ordering ambiguities arise, so that the Dirac bracket computation captures the full quantum operator algebra. We have retained the Dirac bracket nomenclature in order to avoid confusion between commutators of quantum operators and commutators of matrices.

We define the supercharges

$$\begin{aligned} Q_\alpha &= \text{Tr } q_\alpha \\ \tilde{Q}_\alpha &= \text{Tr } \tilde{q}_\alpha \end{aligned} \quad (2.9)$$

where the densities q_α and \tilde{q}_α are $N \times N$ matrices

$$\begin{aligned} q_{\alpha i}^j &= \sqrt{R} \{ P^a \gamma_{\alpha\alpha'}^a + \frac{i}{2} [X^a, X^b] \gamma_{\alpha\alpha'}^{ab}, \theta^{\alpha'} \}_i^j \\ \tilde{q}_{\alpha i}^j &= \frac{2}{\sqrt{R}} \delta_{\alpha\alpha'} \theta_i^{\alpha' j} \end{aligned} \quad (2.10)$$

We want to compute the Dirac brackets in a way which is sensitive to traces of commutators. In principle we should compute the Dirac bracket of densities $\{q_{i\alpha}^j, q_{k\beta}^l\}$. In this computation we encounter terms which are odd under interchange of α and β and also under interchange of the matrix indices. These are analogs of Schwinger terms with odd derivatives of the delta function in field theory charge density commutators. The charges should be defined as large N limits of regularized traces of the densities. As long as we use the same regulator on both charges, these terms antisymmetric in the spinor indices will not contribute to the Dirac bracket of the charges. As a consequence, when computing the DB of the dynamical SUSY charge with itself, we can freely trace on one of the terms in the DB, if we drop pieces of the answer antisymmetric in the spinor indices. This simplifies the computation. After some straightforward but slightly tedious algebra we find:

$$\begin{aligned}
\{\tilde{q}_{i\alpha}^j, \tilde{Q}_\beta\}_{DB} &= \frac{2}{R} \delta_{\alpha\beta} \delta_i^j \\
\{q_{i\alpha}^j, \tilde{Q}_\beta\}_{DB} &= 2(P^a \gamma_{\alpha\beta}^a + \frac{i}{2} [X^a, X^b] \gamma_{\alpha\beta}^{ab})_i^j = 2P_i^{aj} \gamma_{\alpha\beta}^a + 2z_i^{abj} \gamma_{\alpha\beta}^{ab} \\
\{q_{i(\alpha}^j, Q_{\beta)}\}_{DB} &= 4R(\frac{1}{2}P^2 - \frac{1}{4}[X^a, X^b]^2)_i^j \delta_{\alpha\beta} + 2R\gamma_{\alpha\beta}^{abcd} (X^{[a} X^b X^c X^{d]})_i^j \\
&\quad - 2iR\gamma_{\alpha\beta}^b \{P^a, [X^a, X^b]\}_i^j - iR[\theta^{\alpha'}, [X^b, \theta^{\beta'}]] I_{\alpha\beta\alpha'\beta'}^b \\
&= 4Rh_i^j \delta_{\alpha\beta} + 2z_i^{bj} \gamma_{\alpha\beta}^b + 2z_i^{abcdj} \gamma_{\alpha\beta}^{abcd}
\end{aligned} \tag{2.11}$$

where in the last anticommutator we symmetrized over α and β . We used the matrices

$$\begin{aligned}
z^b &= -iR\{P^a, [X^a, X^b]\} - iR[\theta^{\alpha'}, [\theta^{\alpha'}, X^b]] \\
z^{ab} &= \frac{i}{2} [X^a, X^b] \\
z^{abcd} &= RX^{[a} X^b X^c X^{d]}
\end{aligned} \tag{2.12}$$

The matrices

$$\left(\frac{\partial \mathcal{L}}{A_0}\right)_j^j = \Phi_i^j = i[P^a, X^a]_i^j + 2i(\theta^\alpha \theta^\alpha)_i^j \tag{2.13}$$

generate the $SU(N)$ algebra

$$[\Phi_i^j, \Phi_k^l]_{DB} = i(\delta_i^l \Phi_k^j - \delta_k^j \Phi_i^l). \tag{2.14}$$

The Gauss law constraint associated with the gauge $A_0 = 0$ is $\Phi = 0$. Writing

$$z^b = -\{\Phi, X^b\} + i[\{X^b, P^a\}, X^a] + i\{\theta^{\alpha'}, \{\theta^{\alpha'}, X^b\}\} \tag{2.15}$$

shows that for finite N , $\text{Tr } z^b$ vanishes in the space of $SU(N)$ invariant states (satisfying Gauss law).

We interpret $\text{Tr } z^{ab}$ as the two brane charge Z^{ab} and $\text{Tr } z^a$ and $\text{Tr } z^{abcd}$ as the two charges of the wrapped branes Z^a and Z^{abcd} . Note as a consistency check of this interpretation the R dependence of these charges. Z^{ab} is independent of R while Z^a and Z^{abcd} are proportional to R . All these charges are traces of commutators and therefore vanish for finite N . However, for infinite N they can be activated.

de Wit, Hoppe and Nicolai [7], have computed the SUSY charge density algebra on the membrane world volume in light cone supermembrane theory. To compare our results with theirs, substitute $\theta \rightarrow 2^{1/4}\theta$, $q \rightarrow 2^{1/4}q$ and $\tilde{q} \rightarrow 2^{-1/4}\tilde{q}$ in our expressions. Commutators of two bosons $[A, B]$, commutators of a bosons and a fermion $[A, B]$ and anticommutators of two fermions $\{A, B\}$ are all replaced by $-i\epsilon^{rs}\partial_r A \partial_s B$; all other (anti)commutators are simple. Under this transcription our z^{abcd} becomes zero. The longitudinal five brane charge vanishes in the membrane approximation to the matrix model. The authors of [7] also have two terms which are bilinear in θ which do not seem to follow from our computation.

3. BPS Branes in the Matrix Model

As we have seen, light cone supersymmetry breaks up into two 16 component supercharges. Under the kinematical SUSY transformation the fermionic coordinates transform as

$$\delta\theta = \tilde{\epsilon} \quad (3.1)$$

The fermion transformation law under the dynamical SUSY transformations is

$$\delta\theta = \gamma_a P^a \epsilon + \frac{i}{2} [X^a, X^b] \gamma_{ab} \epsilon \quad (3.2)$$

For static solutions, the only way to preserve half of the SUSY generators, is to cancel the kinematical SUSY variation against the dynamical one. The kinematical SUSY transformation changes θ by a multiple of the unit matrix, so we conclude that for static classical BPS configurations of the matrix model, which preserve half of the supersymmetries, the commutators of the X^a must be multiples of the identity:

$$[X^a, X^b] = i\mathcal{F}^{ab} I \quad (3.3)$$

where I is the unit matrix in the $SU(N)$ space. \mathcal{F}^{ab} is antisymmetric, and can always be brought to canonical symplectic (Jordan) form. In [1] it was shown that configurations of this type indeed solve the static classical equations of the matrix model. It is clear that such configurations only exist in the large N limit.

The static classical BPS states of the matrix model are thus characterized by a set of orthogonal transverse two planes. Apart from the transverse membrane solution of [1], we have a four brane, a six brane, and an eight brane. (These names refer only to the transverse dimensions. We will see that the four brane should be interpreted as a five brane wrapped around the longitudinal direction.) We can verify that the membrane solution is indeed the membrane solution of M theory by examining the SUSY algebra. The transverse two brane charge should show up in the anticommutator of kinematical with dynamical SUSY generators. Indeed, as we have seen in the previous section, this anticommutator contains, apart from the transverse momentum generator, the trace of the commutator of two X^a matrices, which is nonzero for the membrane configuration. The calculation of the membrane tension in the appendix of [1] can now be rederived as a BPS formula. Note that this anticommutator should also have contained the transverse five brane charge, but it appears to be absent.

The transverse fourbrane configuration requires four of the X^a to satisfy the commutation relations of two pairs of canonical variables. The uniqueness theorem for irreducible representations of the canonical algebra tells us that we must represent this in the tensor product of two Hilbert spaces. Thus,

$$\begin{aligned} X^1 &\propto q^1 \\ X^2 &\propto p^1 \\ X^3 &\propto q^2 \\ X^4 &\propto p^2. \end{aligned} \tag{3.4}$$

Here, the canonical variables are formal constructions for finite N with commutators proportional to $N^{-\frac{1}{2}}$ (we will explain this scaling below). More generally, if $N = n_1 n_2$, we can have one commutator scale like $\frac{1}{n_1}$ and the other like $\frac{1}{n_2}$. Then, we take $n_1, n_2 \rightarrow \infty$. Reducible representations, corresponding to multiple branes, require a further tensor product with a space in which all of the X^a act as the unit matrix. The four brane clearly carries nonzero values of the transverse two brane charges Z^{12} and Z^{34} , which we constructed in the previous section. We interpret this in the following manner. In the next section we will see that smooth fluctuations around the fourbrane configuration can be viewed as fields in the phase space defined by the canonical variables q^r, p^r . In this field space, the background configuration can be viewed as representing two orthogonal infinite stacks of membranes. The 12 stack is translation invariant in the 34 directions, and *vice versa*.

In addition, the fourbrane configuration has nonvanishing longitudinal five brane charge Z^{1234} . Remember that the corresponding charge density vanishes in the membrane approximation to the matrix model. Thus, the existence of this configuration, like

that of the supergraviton states, is a feature of the matrix model not shared by the membrane Lagrangian. The two kinds of charge are related via $P^+ Z^{1234} = Z^{12} Z^{34}$. This is precisely the relation necessary to ensure the BPS condition for the eleven dimensional SUSY algebra. Thus, we can interpret our transverse four brane as a special configuration of a longitudinal five brane with stacks of two branes in it.

By contrast, the transverse six and eight branes do not seem to correspond to things we expect to find in M theory. M theory compactified on a circle does contain a six brane, but it is a “Kaluza Klein monopole” and would be expected to decouple in the noncompact limit. We know of no seven or eight branes in M theory, and nine branes are supposed to correspond to “ends of the world” which carry gauge dynamics. What then are these configurations which we have found?

Some insight can be gained by calculating the tensions of the various branes. For a brane with $2t$ transverse dimensions, the full space on which the matrices act is a tensor product of t spaces of dimensions n_l ($l = 1...t$). The total dimension is $\prod_l n_l = N$, where $P^+ = N/R$ is the total longitudinal momentum of the system. We take $n_l \rightarrow \infty$ such that $n_l \sim N^{\frac{1}{t}}$. For smooth semiclassical branes, we want to take matrices whose commutators approach Poisson brackets and are interpretable as functions on a $2t$ dimensional phase space. Thus, for the $2t$ brane we want $[X^a, X^b] \sim N^{-\frac{1}{t}}$. The trace on the full vector space approaches N times the integral over phase space. Thus, the tension contribution to the energy of a $2t$ brane scales like

$$\delta E \propto \text{tr} [X^a, X^b]^2 \sim N^{1-\frac{2}{t}} \quad (3.5)$$

For $t = 1$, the membrane, this is proportional to N^{-1} , the appropriate scaling for the energy of a finite object in the infinite momentum frame. For $t = 2$ the energy is constant. This, as we argued in the introduction, is the appropriate behavior for a brane with one dimension wrapped around the longitudinal dimension of the infinite momentum frame.

For branes of arbitrary dimension wrapped around a single circle, we expect to find states of a similar energy, corresponding to longitudinal momentum carried by brane waves which are translationally invariant in the transverse directions. In this case, the energy of the state should be interpreted as a transverse energy density. All of these static solutions of the matrix model correspond to branes wrapped around cycles of a transverse torus, where the size of the torus is encoded in the periodicities implicit in (3.3) and (3.4). Shifts of the X^a by a lattice vector, are gauge transformations. The versions of these constructions appropriate for infinite eleven dimensional space time has the proportionality constants in (3.3) scaled to infinity. All energies scale to infinity in this limit, with finite transverse energy densities. Thus, the brane with 4 transverse dimensions has precisely the energy

density we would expect for a five brane wrapped around the longitudinal direction. This interpretation also fits nicely with the BPS formula, for the solution carries both membrane and fivebrane charges, satisfying the relation which preserves half of the SUSYs.

By contrast, for $t > 2$ the brane's transverse energy density scales to infinity in the IMF. We have not been able to find an interpretation for these BPS states in terms of the conventional M theory menagerie. Perhaps some sense can be made of them after further compactification. Indeed, the computation of the next section shows that their world volume Lagrangians are (at the level of quadratic fluctuations), precisely those of the corresponding Dirichlet branes of perturbative string theory.

To conclude this section we will construct a wrapped fivebrane carrying no membrane charge. To this end, we note that the transformation of the fermionic coordinates under dynamical SUSYs, reduces for static configurations to:

$$\delta\theta = \frac{i}{2}[X^a, X^b]\gamma^{ab}\epsilon \quad (3.6)$$

Half of the variations vanish, if we take only four nonzero X^a which satisfy the condition

$$[X^a, X^b] = \frac{1}{2}\epsilon_{abcd}[X^c, X^d] \quad (3.7)$$

(we raise and lower indices freely using a flat Euclidean metric) where ϵ_{abcd} is the Levi-Civita symbol in the four indices. There are no solutions of this equation for finite matrices. Multiplying by $[X^a, X^b]$ and taking the trace, we recognize the right hand side as our wrapped fivebrane charge, which vanishes by cyclicity of the trace. A large N solution can be obtained by choosing the X^a to be the covariant derivatives in a self-dual Yang-Mills potential:

$$X^a = \frac{1}{i}\frac{\partial}{\partial Q^a} - A_a(Q) \quad (3.8)$$

It was recognized long ago [8] that such configurations are formal solutions of the large N equations for static solutions of our matrix model².

Clearly, the minimally charged fivebrane is one for which we take an $SU(2)$ gauge group. The space on which our large N matrices act is the tensor product of four representations of the canonical commutation relations³ – $(Q^a, P^a(\equiv \frac{1}{i}\frac{\partial}{\partial Q^a}))$ and a doublet

² In this reference the solutions are described as constant classical solutions of the four dimensional Yang Mills equations. M. Douglas and M. Li (private communication) have also investigated this solution in the context of the matrix model of M theory, but with a somewhat different interpretation.

³ Here we mean “truly infinite dimensional matrices,” whose commutator has no inverse power of N in it.

representation of $SU(2)$. This configuration has finite transverse energy density in the IMF, as befits a longitudinal fivebrane.

In conventional Yang-Mills theory, the self-dual configuration with minimal topological charge has five collective coordinates. However, in the present context, the translational collective coordinates are *gauge transformations* of the large N gauge group, generated by the unitary operators $e^{ib_a P^a}$, where the P^a are the canonical variables. The unitary transformation $e^{ic_a Q^a}$ shifts the X^a variables in (3.8) by an arbitrary continuous four vector. As in [1] we interpret this as meaning that the configuration is compactified on a four torus, but now of zero radius. If we compactify the Q^a space on a torus, the X^a matrices are gauge equivalent only to discrete shifts of themselves and we obtain a configuration of the matrix model compactified on the dual torus. This configuration has no scale size collective coordinate (as we explain below). Thus, we obtain a unique longitudinal fivebrane configuration of the matrix model. The uncompactified limit of this configuration is a zero scale size instanton on a torus of zero radius.

There is another, superficially quite different, solution of these equations. Let $[q^m, p^n] = \frac{2\pi i}{N^{1/2}} \delta^{mn}$; $m = 1, 2$ be two pairs of formal finite N canonical pairs such as we employed in the construction of the four brane with two brane charge. Define

$$\begin{aligned} X^1 &= \frac{1}{2}[(p^1 - p^2)\sigma^3 + (p^1 + p^2)] \\ X^2 &= \frac{1}{2}[(q^1 + q^2)\sigma^3 + (q^1 - q^2)] \\ X^3 &= \frac{1}{2}[(p^2 - p^1)\sigma^3 + (p^1 + p^2)] \\ X^4 &= \frac{1}{2}[(q^2 + q^1)\sigma^3 + (q^1 - q^2)] \end{aligned} \tag{3.9}$$

These expressions also define solutions of the BPS condition, with the same energy and charges as (3.8). We have not found a matrix model gauge transformation which maps one into the other. A preliminary analysis of the fluctuations around (3.9), suggests that perhaps this configuration can be separated to two disjoint objects. We do not know how to interpret this fact.

3.1. Compactification on a Four Torus

Another view of these configurations is obtained by compactifying the system on a four torus⁴. As argued in [1,3,4], the relevant variables of the matrix model are then most

⁴ This section was motivated by remarks of L. Susskind, who first noticed that the wrapped five brane was an instanton.

elegantly packaged as the dimensional reduction of ten dimensional SYM theory on the dual four torus (though this description makes certain assumptions about how the large N limit is to be taken). The original zero branes and the strings connecting them are described by the SYM variables. The membrane wrapped around a two torus is a toron [9]. In this description the two brane charge and the membrane are visible already for finite N but the charge is conserved modulo N . Finally, a longitudinal fivebrane wrapped around the four torus becomes a point object in this description – an instanton.

In string theory, we can consider the point fivebrane sitting at some transverse distance from the collection of zero branes (which look like transverse D4-branes in this description). The relative transverse position is described by Higgs fields in the fundamental representation of the large N gauge group. If the matrix model of M theory is complete, then this must be equivalent to some configuration of the zero branes themselves. Indeed, at large N , the quantum fluctuations of the Higgs degrees of freedom are negligible. The fivebrane is completely described by the classical “image” it makes in the $4 + 1$ dimensional SYM theory, which is well known to be an instanton [10]. We conclude that on a 4 torus, the correct description of the wrapped fivebrane is as an instanton solution of the $4 + 1$ dimensional gauge theory on the dual torus (which of course is a soliton in this context). On a finite torus, the scale of an instanton is not a collective coordinate, but instead is determined. The action is bounded below by the topological charge, and this bound is achieved by the zero size instanton. Note that in the decompactification limit, the dual torus shrinks to zero radius, so we would not expect to regain the collective coordinate for the scale of the instanton.

This compactified picture also sheds some light on the fivebranes with two brane charge which we discussed above. SYM theory on a torus has toron configurations [9] which carry topological charge equal to the product of Z_N fluxes in orthogonal planes. Configurations whose topological charge derives solely from Z_N flux have twice as much SUSY as instantons, and correspond to the wrapped fivebranes with embedded two branes which we described above.

We can give a unified description of all the branes we discussed in the framework of the compactified SYM theory. We use $\pi_{2k-1}(U(N)) = Z$ for $N \geq k$, which is measured by $Q_k = \int \text{tr } F^k$. Then Q_k is a $2k$ brane charge; i.e. a gauge configuration with non-zero Q_k is a $2k$ brane.

We would like to emphasize that a comparison of the discussion of the present subsection with that immediately preceding it shows once again that all of the physics of the compactified theory is completely encoded in the large N matrix quantum mechanics with no additional degrees of freedom.

4. Fluctuating Branes

In this section we study the small fluctuations around our BPS configurations. For simplicity, we consider only the bosonic degrees of freedom. The fermions can easily be added.

We have shown that BPS solutions preserving 16 linear combinations of the kinematical and dynamical SUSYs satisfy.

$$[X^a, X^b] = i\mathcal{F}^{ab}I \quad (4.1)$$

where \mathcal{F}^{ab} are numbers and I is the unit matrix, Using the $SO(9)$ symmetry we can bring C to a Jordan canonical form $\mathcal{F}^{12} = -\mathcal{F}^{21}$, $\mathcal{F}^{34} = -\mathcal{F}^{43}$, $\mathcal{F}^{56} = -\mathcal{F}^{65}$, $\mathcal{F}^{78} = -\mathcal{F}^{87}$ and all others vanish.

We replace the indices a, b by $r, s = 1, \dots, n$ and $I = n+1, \dots, 9$ such that $\mathcal{F}^{IJ} = \mathcal{F}^{rI} = 0$ and again freely raise and lower indices using the flat Euclidean metric. We expand the matrix variables around the classical solution $X^r = U_r$, $X^I = 0$

$$X^r = U_r + A_r. \quad (4.2)$$

We define the “field strength”

$$\begin{aligned} F_{0r} &= -F_{r0} = \partial_0 A_r + i[U_r, A_0] \\ F_{rs} &= -i[U_r, A_s] + i[U_s, A_r] \end{aligned} \quad (4.3)$$

which is gauge invariant under

$$\begin{aligned} \delta A_0 &= \partial_0 \lambda \\ \delta A_r &= -i[U_r, \lambda]. \end{aligned} \quad (4.4)$$

This gauge transformation is the linearized version of the $U(N)$ gauge transformation

$$\begin{aligned} \delta A_0 &= \partial_0 \lambda + i[\lambda, A_0] \approx \partial_0 \lambda \\ \delta A_r &= -i[U_r, \lambda] + i[\lambda, A_r] \approx -i[U_r, \lambda] \\ \delta X^I &= i[\lambda, X^I] \approx 0. \end{aligned} \quad (4.5)$$

The field strength satisfies the “Bianchi identity”

$$\partial_0 F_{rs} - i[U_r, F_{s0}] - i[U_s, F_{0r}] = 0. \quad (4.6)$$

Both in verifying gauge invariance under (4.4) and in checking (4.6) one must use the Jacobi identity and the fact that $[U_r, U_s]$ is proportional to the unit matrix.

We expand the bosonic terms in the Lagrangian to quadratic order in X^I, A_0, A_r and we drop traces of commutators except $\text{Tr} [U_r, U_s] = iN\mathcal{F}^{rs}$

$$\begin{aligned} & \text{Tr} \left(\frac{1}{2}(D_0 X^I)^2 + \frac{1}{2}(D_0 X^r)^2 + \frac{1}{2}[X^r, X^I]^2 + \frac{1}{4}[X^r, X^s]^2 + \frac{1}{4}[X^I, X^J]^2 \right) \\ & \approx \text{Tr} \left(\frac{1}{4}F_{0r}^2 + \frac{1}{4}F_{r0}^2 - \frac{1}{4}F_{rs}^2 + \frac{1}{2}(\partial_0 X^I)^2 + \frac{1}{2}[U_r, X^I]^2 - \frac{1}{4}(\mathcal{F}^{rs})^2 \right). \end{aligned} \quad (4.7)$$

For the special case that only $\mathcal{F}^{12} = -\mathcal{F}^{21} \neq 0$ ($n = 2$) we can dualize the gauge field A_0, A_r to a single gauge invariant field ϕ

$$\begin{aligned} F_{12} &= \partial_0 \phi \\ F_{0r} &= -i[U_s, \phi]\epsilon_{rs}. \end{aligned} \quad (4.8)$$

Under this substitution the equation of motion of F is satisfied trivially while the Bianchi identity of F is ensured by the equation of motion from the new Lagrangian

$$\frac{1}{2}\text{Tr} \left((\partial_0 \phi)^2 + (\partial_0 X^I)^2 + [U_r, X^I]^2 + [U_r, \phi]^2 - (\mathcal{F}^{12})^2 \right). \quad (4.9)$$

Returning to the general case $r = 1, \dots, n$, we can now view U_r as coordinates on an n dimensional phase space. Then, the matrices X^I, A_0, A_r can be replaced by functions on this phase space. The commutators with U_r become derivatives

$$-i[U_r, \mathcal{O}] = \partial_r \mathcal{O} \quad (4.10)$$

for any function on phase space \mathcal{O} . Note that the fact that the U_r 's do not commute is consistent with the fact that derivatives commute:

$$\partial_r \partial_s \mathcal{O} - \partial_s \partial_r \mathcal{O} = -[U_r, [U_s, \mathcal{O}]] + [U_s, [U_r, \mathcal{O}]] = [\mathcal{O}, [U_r, U_s]] = 0 \quad (4.11)$$

where we again used the Jacobi identity and the fact that $[U_r, U_s]$ is proportional to the unit matrix.

This turns (4.7) to a n dimensional Lorentz covariant Lagrangian including a $U(1)$ gauge field and $9-n$ scalars. This is the Lagrangian describing Dn -branes in the IIA string theory. For the special case of 2-branes, the dual variable ϕ corresponds to the eleventh dimension.

Thus, we have verified that the small fluctuations around these BPS solutions of the matrix model contain the collective coordinates of branes of appropriate dimension, which justifies our name for these objects. It would be interesting to repeat these calculations for the longitudinal fivebranes with no membrane charge.

We can also verify the existence of enhanced gauge symmetries when two or more parallel membranes of the same charge are brought together. In doing so it is important to remember that in the above calculations we have been using the Poisson bracket approximation for the commutators of matrices which are functions of the canonical variables which define the membrane background. When M parallel membranes sit at the same point in the transverse dimensions, the classical solution becomes the above single membrane solution tensored with the unit matrix in an M dimensional space. This is an example of the block diagonal construction of multi-extended-object states which was described in [1]. When we consider small fluctuations around the classical configuration, they are described by general hermitian matrices in this M dimensional space. The commutator of the two coordinates in the membrane direction is, in the Poisson bracket approximation,

$$[X^1, X^2] = \partial_1 A_2 - \partial_2 A_1 + i[[A_1, A_2]], \quad (4.12)$$

where A_r is defined as above, but is now a matrix valued field. The double bracket symbol $[[A, B]]$ refers to commutators of $M \times M$ matrices. It is easy to see that the electric field strength and gauge transformation laws described above also generalize to the appropriate $U(M)$ covariant formulae. The terms involving transverse coordinates become the covariant derivatives for Higgs fields in the adjoint of $U(M)$. Thus, in the Poisson bracket approximation, the multimembrane system has a $U(M)$ enhanced gauge symmetry when the membrane positions coincide.

Note that in the full matrix model, the ultraviolet properties of this gauge theory will be regularized by the noncommutative geometry of the membrane volume. However, its infrared behavior is unaffected. Since the gauge coupling is a relevant operator the theory evolves to strong coupling in the infrared. The moduli space of vacua of the theory is the Cartan subalgebra of the Lie algebra of the gauge theory, modded out by the Weyl group. It is possible that the theory at the origin is a free field theory with the Weyl group acting as a gauge symmetry (an orbifold). Alternatively, the theory there could be at a non-trivial infrared fixed point of the renormalization group. Either way, the off-diagonal gauge bosons of the gauge group are not the right degrees of freedom at long distance. The matrix model tells us nothing new about these possibilities. It only shows us that the phenomenon of enhanced gauge symmetry for coinciding branes occurs in M theory, just as it does for D-branes in perturbative string theory⁵.

⁵ For membranes in flat 11 dimensional space this gauge theory is at infinite coupling and therefore there should not be any remnant of the off diagonal gauge bosons (of course, they are important upon compactification).

5. Conclusions

The SUSY algebra of the matrix model contains an enormous amount of information and has led us to the discovery of a class of p-brane solitons including the longitudinal fivebrane predicted by M theory. This makes it all the more striking that there is no vestige of the transverse fivebrane charge in the matrix model. This is incompatible with Lorentz invariance, so it is clear that there is something missing in the rules for the matrix model which have been elaborated up to this point. We emphasize that this charge should have appeared in the “easy” anticommutator between the kinematical and dynamical SUSY generators. Although we can imagine adding “improvement terms” to the SUSY generators which generate terms with the right tensor structure, they are somewhat arbitrary and have not as yet shed any light on the puzzle.

There are two logically separate issues:

1. Does the finite N matrix model contain states which become transverse fivebranes in the large N limit?
2. What modifications of the matrix model rules must we make in order to incorporate the fivebrane charge into the matrix model SUSY algebra?

At the moment we have two sets of clues which appear to give somewhat contradictory answers to the first of these questions. The first arises in the context of compactification. As currently understood, the rules for torus compactification of the matrix model lead to large N SYM theory on the dual torus. In the case of a transverse three torus, field theoretic S-duality leads [4] to a description of the wrapped five brane in terms of a classical configuration of the dual Higgs fields of the SYM theory. This suggests that the matrix model contains the transverse fivebrane, but that it is not a classical configuration of the original matrix variables. It seems extremely important to find a more explicit construction of this configuration and to decompactify it.

On the other hand, there is one well known fact about perturbative string theory which suggests that the puzzle of transverse five branes may be an artifact of our enforced reliance on a light cone gauge description of the matrix model. *There are no transverse D-branes in light cone gauge string theory.* Transverse D-branes are not longitudinally translation invariant and when boosted to the IMF, they are no longer static objects⁶. Note that large transverse strings *are* allowed in the light cone formalism, because strings

⁶ More formally, the Virasoro condition, $\partial_s X^- = \partial_s X^a \partial_t X^a$ (where s and t are world sheet space and time coordinates) shows that if we choose either Dirichlet or Neumann boundary conditions for the transverse variables, the longitudinal coordinate always satisfies Neumann boundary conditions.

are not D-branes. A parallel can be made between these observations and our discovery of transverse membranes, but not five branes, in the matrix model. Membranes appear to be the elementary objects of the matrix model, while fivebranes are expected to be D branes [11]. This comforting analogy suggests that the missing five brane charge might be only a gauge artifact, but it fails to account for the existence of the wrapped transverse fivebrane of [4] .

We are thus left with an unresolved puzzle. The current formulation of the matrix model may well contain dynamical fivebranes and admit a covariant generalization. Alternatively, a fully covariant, nonperturbative formulation of M theory may have to introduce membranes and five branes on an equal footing as elementary objects. The resolution of this conundrum is of the utmost importance.

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